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THE COST-OF-CARRY FORMULA TO DETERMINE FUTURES PRICES: HOW WRONG CAN YOU BE?*

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ABSTRACT

In this note we analyze the pricing error of using the cost-of-carry formula to determine futures prices. When the underlying asset is a share of stock, the sign of the pricing error is basically determined by the sign of the correlation between the stock return and short-term risk-free interest rate. When the underlying asset is a zero-coupon bond, the forward price is an upper bound for the pricing error.

Keywords: Cost-of-carry Formula; Forward Prices; Futures Prices; Pricing Error.
JEL Classification: G12, G13.

INTRODUCTION

It is well known, if interest rates are deterministic, that forward- and futures prices coincide, see e.g., Cox, Ingersoll, and Ross (1981). However, there is plenty of empirical evidence showing that interest rates evolve randomly over time, possibly causing forward- and futures prices to differ.

Forward prices can (in theory) easily be determined from the prices of traded assets using the cost-of-carry formula. In many, if not most introductory textbooks in finance, the cost-of-carry formula is also used to determine futures prices. The question we like to address in this note is *"How wrong can you be if you apply the cost-of-carry formula to determine futures prices?"*. This note contributes to the litterature on futures contracts in two ways. First, we derive error bounds for the futures price when it is estimated by the cost-of-carry formula. These error bounds indicate that the pricing errors can be large. Secondly, we illustrate with two numerical examples using realistic parameter values that the pricing errors are likely to be small.

ECONOMIC SET-UP

We assume a complete and arbitrage-free market where trading takes place in continuous time. Modulo technical conditions, there then exists an equivalent martingale measure Q where price processes of non-dividend paying assets are martingales when discounted with the bank account. For analytical simplicity, we assume that the evolution of interest rates can be described by a Gaussian version of the Heath, Jarrow, and Morton framework (Heath, Jarrow, and Morton (1992)). The short-term risk-free interest rate at some future point in time s (under Q) is then given by

$$r_s = f(t,s) + \int_t^s \sigma_f(v,s)^{\mathrm{T}} \int_v^s \sigma_f(v,u)\,du\,dv + \int_t^s \sigma_f(v,s)^{\mathrm{T}} d\mathbf{W}_v,$$

where $f(t,s)$ is the instantaneous forward rate for time s prevailing at time t, $\sigma_f(t,s)$ is a two-dimensional vector of deterministic volatility functions, and $\mathbf{W}$ is a two-dimensional vector of

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standard Brownian motions. The accumulated return on the bank account over some time period t to s is

$$\beta_{t,s} = \int_t^s r_v dv = -\ln B(t,s) + \frac{1}{2}\sigma_\beta^2 + \int_t^s (\int_v^s \sigma_f(v,u)du)^{\mathrm{T}} d\mathbf{W}_v,$$

where $B(t,s)$ is the time t value of a zero-coupon bond maturing at time s and $\sigma_\beta^2 = \int_t^s \| \int_v^s \sigma_f(v,u)du \|^2 dv$. A bank account with time t deposit M_t will at time s have value $M_s = M_t e^{\beta_{t,s}}$. In the following we set $M_t = 1$.

We consider two underlying assets; a non-dividend paying stock with price S_t and a zero-coupon bond maturing at time U with price $B(t,U)$. Under the equivalent martingale measure Q, the time s prices, $t < s < U$, are given by

$$S_s = S_t e^{\int_t^s (r_v - \frac{1}{2}\|\sigma(v)\|^2)dv + \int_t^s \sigma(v)^{\mathrm{T}} d\mathbf{W}_v},$$

where $\sigma(v)$ is a two-dimensional vector of deterministic volatility functions and

$$B(s,U) = B(t,U) e^{\int_t^s (r_v - \frac{1}{2}\int_v^U \|\sigma_f(v,u)du\|^2)dv - \int_t^s \int_v^U \sigma_f(v,u)^{\mathrm{T}} du d\mathbf{W}_v}.$$

COST-OF-CARRY FORMULA AND THE FUTURES PRICE

Let $F_{t,T}^Z$ be the forward price you can agree upon at time t for delivery at time T and $f_{t,T}^Z$ the corresponding futures price. It is well known (see e.g., Cox et al. (1981) and Musiela and Rutkowski (1997)) that

$$f_{t,T}^Z = E_t^Q[Z_T]$$

and

$$F_{t,T}^Z = E_t^{Q_T}[Z_T] = \frac{Z_t}{B(t,T)},$$

the cost-of-carry formula. Here E_t is the conditional expectation operator, Z is the spot price of the underlying asset, and Q_T is a probability measure equivalent to Q and is determined by the Radon-Nikodym derivative

$$\frac{dQ_T}{dQ} = \frac{M_T^{-1}}{B(t,T)} = e^{-\frac{1}{2}\sigma_\beta^2 - \int_t^T (\int_v^T \sigma_f(v,u)du)^{\mathrm{T}} d\mathbf{W}_v}.$$

With the assumed price processes in this note, the futures price for the stock is

$$f_{t,T}^S = \frac{S_0}{B(0,T)} e^{\sigma_\beta^2 + \int_t^T \sigma(v)^{\mathrm{T}} \int_v^T \sigma_f(v,u)dudv}$$

$$= F_{t,T}^S e^{\sigma_\beta^2 + \int_t^T \sigma(v)^{\mathrm{T}} \int_v^T \sigma_f(v,u)dudv}. \tag{1}$$

The futures price is equal to the forward price, multiplied by a correction for the variance of accumulated short-term interest rates (σ_β^2) and the covariance between returns on the underlying asset and the bank account. We have deterministic interest rates when $\sigma_f(t,s)=\mathbf{0}$, and (1) then implies that $f_{t,T}^S = F_{t,T}^S$. The two prices also coincide if

$$\sigma_\beta^2 = -\int_t^T \sigma(v)^{\mathrm{T}} \int_v^T \sigma_f(v,u)dudv.$$

This equality can only hold if the returns, under Q, on the underlying asset in excess of the return on the bank account and the return on the bank account are negatively correlated. The corresponding futures price for the zero-coupon bond maturing at time $U > T$ is

$$f_{t,T}^B = F_{t,T}^B e^{-\int_t^T \int_v^T \sigma_f(v,u)^{\mathrm{T}} du \int_T^U \sigma_f(v,u)dudv}. \tag{2}$$

To better get an understanding of the quantitative difference between forward- and futures prices, we use a specification of the volatility functions that has sometimes been used in the literature (see e.g., Miltersen and Persson (1999)). We assume that

$$\sigma(t) = \sigma_S \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

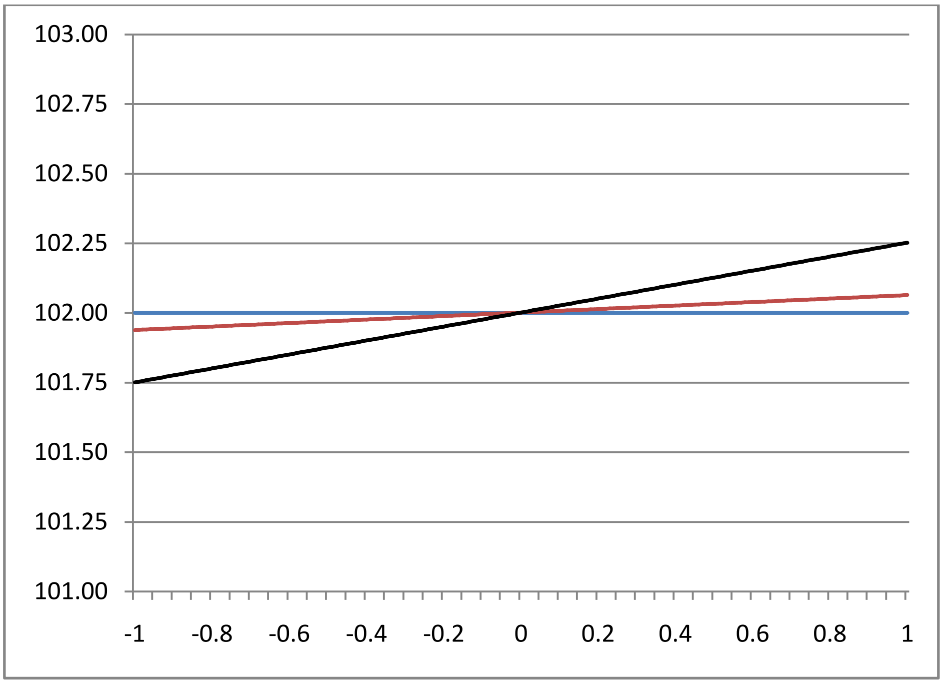
$$\sigma_f(v,u) = \sigma e^{-\kappa(u-v)} \begin{bmatrix} \varphi \\ \sqrt{1-\varphi^2} \end{bmatrix},$$

where σ_S, σ, κ, and φ are constants. Let $h(T)$ be the exponent in (1):

$$\begin{aligned} h(T) &= \frac{\sigma^2}{2\kappa^3}(2\kappa(T-t)-3+4e^{-\kappa(T-t)}-e^{-2\kappa(T-t)}) \\ &+\frac{\sigma_S\sigma\varphi}{\kappa^2}(\kappa(T-t)-1+e^{-\kappa(T-t)}). \end{aligned}$$

The first term of $h(T)$ is σ_β^2 and is always non-negative. For $\varphi > 0$, also the second term is non-negative. To see this, consider $f(x) = x-1+e^{-x}$, $x \ge 0$ (κ is the force-of-gravitation in a mean-reverting process and is positive). We note that $f(0)=0$ and $f'(x)=1-e^{-x}>0$ for $x>0$. The function $f(x)$ is therefore positive. Similarly, because σ_S and σ are non-negative, the second term of $h(T)$ has the same sign as φ. Let $\sigma_S = L$, where L is “large". For $\varphi < 0$, the forward price $F_{t,T}$ is an upper bound for the error in the futures price by using the cost-of-carry formula. For $\varphi > 0$, there is no upper bound for the error in the sense that, *ceteris paribus*, $\sigma_S = 2L$ gives a larger error than $\sigma_S = L$. In practice, both σ_S and $T-t$ have moderate values (typically less than 1), and the pricing errors thereby become relatively small. We illustrate the pricing error with a numerical example. Assume that the futures contract matures in six months ($T-t=0.5$) and that the initial term structure of interest rates is flat and equal to 3.96%. With $S_t = 100$, this term structure gives a forward price of $F=102$. We further assume $\kappa = 0.1$. In figure 1 we plot futures prices as a function of φ for the two cases $\sigma_S = 0.5$, $\sigma = 0.04$ and $\sigma_S = 0.25$, $\sigma = 0.02$. The forward price is also plotted for reference (the horizontal line). Clearly, the model predicts a fairly modest pricing error.

FIGURE 1



The figure shows the forward price (horizontal line at 102) and futures prices for different choices of φ. Parameter values are $S_t = 100$, $r = 0.0396$, $\kappa = 0.1$, and $T - t = 0.5$. The set of futures prices with the steepest inclination is for $\sigma_S = 0.5$ and $\sigma = 0.04$, while the second set of futures prices is for $\sigma_S = 0.25$ and $\sigma = 0.02$

There are no calendar effects in the interest rate model used in this note, and, to somewhat simplify expressions, we can therefore without loss of generality set $t = 0$. Let $g(U)$ be the *integrand* of the outer integral in the exponent in (2):

$$g(U) = \frac{\sigma^2}{\kappa^2}(e^{-\kappa(T-v)} - e^{-\kappa(U-v)} - e^{-\kappa(T-v)} + e^{-\kappa(U+T-2v)}).$$

We have that $g(T) = 0$. Furthermore,

$$\frac{dg(U)}{dU} = \frac{\sigma^2}{\kappa} e^{-\kappa(U-v)}(1 - e^{-\kappa(T-v)}) \geq 0 \quad for\ 0 \leq v \leq T.$$

Thus, the integrand is positive (0 for $v = 0$) in the area of integration and the *exponent* in (2) is negative. Futures prices are therefore lower than forward prices and the forward price is an upper bound for the pricing error. This observation is in line with the theoretical discussion in [?] (see his page 315). In figure 2 we plot futures prices as a function of the difference between U and T for two cases, $\sigma = 0.02$ and $\sigma = 0.1$ (forward prices are also plotted, but they cannot be distinguished from the futures prices for the case $\sigma = 0.02$). For realistic parameter values, the error made by using the cost-of-carry formula to determine futures prices is small.

FIGURE 2

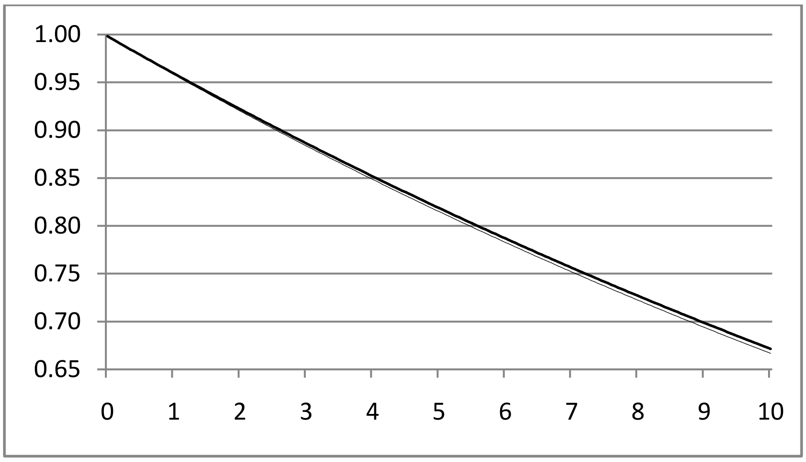


Figure 2: The figure shows futures prices for different choices of $U - T$. Parameter values are $r = 0.0396$, $\kappa = 0.1$, and $T - t = 0.5$. The highest futures prices (dotted line) are for $\sigma = 0.02$ and the lowest for $\sigma = 0.1$. Forward prices are higher than the futures prices, but in the figure they cannot be distinguished from the highest futures prices.

Factors that may be relevant for explaining any difference between forward- and futures prices that we have excluded from the analysis here include transaction costs, collateral, and counterparty risk.

CONCLUSIONS

In this note we analyze futures prices in an economy with stochastic interest rates. We use an asset-pricing model to derive error bounds for the futures price when it is estimated by the cost-of-carry formula. The error bounds are not tight, implying that the pricing errors can be large. The magnitude of the errors depends on the parameters of the model. However, realistic parameter values in combination with the short time untill exparation for futures contracts (often less than three or six months), lead to modest pricing errors. We illustrate this fact with numerical examples for futures contracts on a non-dividend paying stock and on a zero-coupon bond.

REFERENCES

Cox, J., Ingersoll, J., & Ross, S. (1981). The Relation between Forward Prices and Future Prices, *Journal of Financial Economics , 9* (4), 321–346.

French, K. R. (1983). A Comparison of Futures and Forward Prices, *Journal of Financial Economics , 12* (3), 311–342.

Heath, D., Jarrow, R., & Morton, A. (1992). Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation, *Econometrica , 25* (1), 77–106.

Miltersen, K. & Persson, S. A. (1999). Pricing Rate of Return Guaranteesin a Heath-Jarrow-Morton Framework, *Insurance: Mathematics and Economics , 25* (3), 307–326.

Musiela, M. & Rutkowski, M. (1997). *Martingale Methods in FinancialModeling.* Springer Verlag, Berlin Heidelberg.